

Two-loop Feynman integrals for ϕ^4 theory with long-range correlated disorderM. Dudka¹*Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine,
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Two-loop massive Feynman integrals for ϕ^4 field-theoretical model with long-range correlated disorder are considered. Massive integrals for the vertex function $\Gamma^{(4)}$ including two or three massless propagators for generic space dimension and for any value of the correlation parameter are evaluated analytically applying Mellin-Barnes method as well as familiar representation for one-loop integrals. Obtained expressions are presented in the form of hypergeometric functions.

I. INTRODUCTION

Feynman integrals are known to appear in various branches of theoretical physics exploiting perturbative quantum field theories. Recent interest to these objects in high energy physics is generated by necessity of evaluation of radiative corrections (expressed in terms of complicated Feynman integrals) to compare analytical results with experimental data of Large Hadron Collider. To achieve satisfactory accuracy it requires exploitation of high-loop order integrals. However even evaluation of certain low-order Feynman integrals is not a trivial task since there is no unique receipt for calculation of every Feynman integral. Rather Feynman integrals can be successfully evaluated combining different methods¹.

Feynman integrals are frequently used also in statistical physics, where the quantum field theory is exploited for the critical phenomena description. There Feynman integrals are involved in different renormalization group (RG) schemes² for calculation of physical observables, like critical exponents, amplitude ratios, scaling functions, *etc.* Being universal quantities, these observables depend only on global characteristics of a system, like dimension of space d , order parameter dimension N , internal symmetries. One of the fundamental problems there is dependence of the results for general space dimension d .

Usually such dependence can be studied within famous Wilson-Fisher expansion³ in deviation ϵ of a space dimension from its upper critical value. $O(N)$ symmetric ϕ^4 theory has 4 as an upper critical dimension and thus $\epsilon = 4 - d$. High orders of the ϵ -expansion allow to obtain accurate values for critical exponents of three-dimensional $O(N)$ models⁴. Calculation of critical exponents within massive field theory directly at fixed space dimension $d = 2$ or $d = 3$ proposed by Parisi⁵ serves as an alternative to this method. To apply such approach one needs to know the values of massive Feynman integrals for given d within considered loop number. Estimates of critical exponents on the base of this method for the three-dimensional ϕ^4 theory are in correspondence with the values given by ϵ -expansion⁴. This method was successfully used to study modifications of ϕ^4 model for three-dimensional systems with cubic anisotropies⁶⁻⁸, uncorrelated weak quenched disorder⁸⁻¹⁶, multicriticality¹⁷, random anisotropies^{15,18-21}, frustrations^{8,22-24}. Note that these modifications do not change structure of massive Feynman integrals for the ϕ^4 theory.

Fixed space dimension approach can be also used for studies at non-integer values of d . Thus numerical calculation of massive integrals of ϕ^4 theory for general dimension $2 < d < 4$

was performed within two-loop order in a study of disordered Ising system²⁵. The numerical values for three-loop massive integrals were later obtained for $0 < d < 4$ ²⁶ and used for analysis of critical properties in random Ising in space dimension range $2 < d < 4$ ²⁷. Recently dependence of two-loop massive integrals of $O(N)$ symmetric ϕ^4 field theory was calculated analytically obtaining integrals in a compact form of Gauss hypergeometric functions with d -dependent parameters²⁸.

However field-theoretical description of some complex systems may include also modified integrals. For instance, introduction of an anisotropy to interactions as in a system with Lifshitz critical point²⁹ results in Feynman integrals with different masses in propagators. Recent calculations of corresponding one-loop massive integral for general d present results in form of hypergeometric Appell functions as well as new reduction for them³⁰. Anisotropy in correlation also appears for models of magnets with non-magnetic defects that are correlated in ϵ_d dimension and randomly distributed in $d - \epsilon_d$ space^{31–36}. Presence of such defects also changes the structure of Feynman integrals.

Here we are interested in another kind of correlated disorder, where correlations between defects decay at large distance R between them according to power law R^{-a} . As it is shown below, massive Feynman integrals in this case may have additional massless propagators. Effects of such long-range correlated disorder on the critical properties of ϕ^4 model were studied intensively including static critical behaviour^{37–41}, critical dynamics near equilibrium^{38,39,41,42}, short time critical dynamics^{43–45}, critical ultrasound propagation⁴⁶. Disorder with long-range correlations appears in systems of different nature. Studies the phase transition in superconductor with long-range correlated impurities⁴⁷, statics and dynamics of elastic systems in disordered media^{48,49}, conformal properties of polymers in disordered environment^{15,50}, quantum critical behaviour in systems with long-range correlated impurities⁵¹, percolation in correlated systems^{52,53} can be mentioned. Recently two-dimensional fermionic system with long-range correlated disorder relevant for description of disordered graphene was investigated⁵⁴.

In this paper, analytical calculations of two-loop massive Feynman integrals for systems with long range correlated disorder are performed for general space dimension d and correlation parameter a . Such integrals were known analytically only in one-loop order for $d = 3$ and general a , in two-loop order they were calculated numerically for $d = 3$ and $2 < a < 3$ ^{38,39}. A challenge of this paper is to obtain expressions for two-loop integrals for

general space dimension d and correlation parameter a via known functions. The set-up of the paper is the following. In Section II field theoretical description for systems with correlated disorder as well as two-loop integrals appearing within such description are presented. In the next Section III the one loop expressions are considered integration over internal momentum was performed. In the Section IV results for two-loop integrals are presented in form of hypergeometric functions with d - and a - dependent parameters. Section V summarizes the paper. Definitions of functions as well as some intermediate calculations are given in the Appendices.

II. FIELD-THEORETICAL MODEL WITH CORRELATED DISORDER

We consider system with quenched defects having correlation function $g(R)$ dependent on the distance R between them. To deal with quenched disorder one should average free energy over disorder configurations. It can be performed with help of replica trick, that gives effective Hamiltonian of ϕ^4 type³⁷:

$$\mathcal{H} = \sum_{\alpha=1}^n \int d^d R \left[\frac{1}{2} \left(r_0 \phi_\alpha^2 + (\nabla \phi_\alpha)^2 + \frac{u_0}{4!} (\phi_\alpha^2)^2 \right) \right] - \sum_{\alpha, \beta=1}^n \int d^d R d^d R' g(|R - R'|) \phi_\alpha^2(R) \phi_\beta^2(R') \quad (1)$$

Here, ϕ_α is N -component vector. In this field-theoretical model parameter r_0 is a linear function in temperature and plays a role of a bare mass, $u_0 > 0$ corresponds to bare coupling of $O(N)$ -symmetrical model. The long-distance properties of (1) in the replica limit $n \rightarrow 0$ describe critical behaviour occurring in the disordered system involving correlations between defects. In the model considered here, disorder correlations weaken according to the power law $g(R) \sim R^{-a}$ for large separation R ³⁷. Furier transform of the correlation function for defects gives

$$\bar{g}(\mathbf{k}) = v_0 + w_0 k^{a-d} \quad (2)$$

for small k . In the case $a > d$ the second term of $\bar{g}(\mathbf{k})$ is irrelevant for $k \rightarrow 0$ and therefore this case corresponds to short range disorder. We are interested in the case when $a < d$ that makes term with w coupling crucial at small k . On of he interpretation of the model is that disorder correlation function with $a = d - 1$ corresponds to the case of straight lines of impurities with random orientation, while case $a = d - 2$ corresponds to random planes of impurities.

Standard tool to describe critical behavior is application of field theoretical renormalization group approach². In this approach, the vertex functions $\Gamma^{(n)}$ are considered, their finiteness is ensured by imposing certain normalisation conditions. Using the renormalization at fixed mass and zero external momenta⁵ one has to calculate massive Feynman integrals involving momentum integration of dimension d . Feynman integrals for the Hamiltonian (1) with (2) appearing within two-loop approximation were presented in form of Feynman diagrams together with diagrammatic rules in³⁹. In general case these integrals can be written as:

$$I(\alpha, \beta, \gamma, \delta, \rho, \tau) = \int_{q_1} \int_{q_2} \frac{(q_1^\alpha q_2^\beta |\mathbf{q}_1 + \mathbf{q}_2|^\gamma)^{a-d}}{(r + q_1^2)^\delta (r + q_2^2)^\rho (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\tau}, \quad (3)$$

where we omit in the notation dependence on r , a and d for simplicity. Integration in (3) means:

$$\int_q \{\dots\} = \frac{1}{(2\pi)^d} \int d\mathbf{q} \{\dots\} = \frac{1}{(2\pi)^d} \prod_{i=1}^{d-2} \int_0^\pi d\theta_i \sin^i \theta_i \int_0^{2\pi} d\varphi \int_0^\infty q^{d-1} dq \{\dots\}. \quad (4)$$

Numerators in (3) appear for diagrams including renormalized coupling w only. At $a = d$ we get in (3) a form of two-loop integrals relevant to usual ϕ^4 theory².

Calculations of Feynman diagrams with one w -vertex (with one momentum in the numerator on (3)) can be simply performed by standard integration methods, while evaluation of Feynman integrals with two or three w -vertices presents more difficult task. Here, calculating

four-point vertex function $\Gamma^{(4)}(k_i, r, u_i)$ one meets nine such integrals:

$$I(1, 1, 0, 2, 0, 2) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)}}{(r + q_1^2)^2 (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^2}, \quad (5)$$

$$I(1, 1, 0, 3, 0, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)}}{(r + q_1^2)^3 (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}, \quad (6)$$

$$I(1, 1, 0, 2, 1, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)}}{(r + q_1^2)^2 (r + q_2^2) (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}, \quad (7)$$

$$I(1, 1, 1, 2, 0, 2) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)} |\mathbf{q}_1 + \mathbf{q}_2|^{(a-d)}}{(r + q_1^2)^2 (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^2}, \quad (8)$$

$$I(2, 1, 0, 3, 0, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{2(a-d)} q_2^{(a-d)}}{(r + q_1^2)^3 (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}, \quad (9)$$

$$I(2, 1, 0, 2, 1, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{2(a-d)} q_2^{(a-d)}}{(r + q_1^2)^2 (r + q_2^2) (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}, \quad (10)$$

$$I(1, 2, 0, 2, 1, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{2(a-d)}}{(r + q_1^2)^2 (r + q_2^2) (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}, \quad (11)$$

$$I(1, 1, 0, 1, 1, 2) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)}}{(r + q_1^2) (r + q_2^2) (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^2}, \quad (12)$$

$$I(1, 1, 1, 2, 1, 1) = \int_{q_1} \int_{q_2} \frac{q_1^{(a-d)} q_2^{(a-d)} |\mathbf{q}_1 + \mathbf{q}_2|^{(a-d)}}{(r + q_1^2)^2 (r + q_2^2) (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)}. \quad (13)$$

Main goal of this paper is evaluation of these nine integrals. It is done in the next two sections.

III. EVALUATION OF INTERNAL INTEGRALS

To calculate integrals (5)-(13) presented above we perform subsequent integration over two internal momenta. Let us first perform an integration over q_2 . As one can see the integrals over q_2 for (5)-(13) can be represented as four q_1 -dependent functions:

$$f_1(\alpha, \beta, q_1) = \int_{q_2} \frac{1}{(q_2^2)^\alpha (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\beta}, \quad (14)$$

$$f_2(\alpha, \beta, q_1) = \int_{q_2} \frac{1}{(q_2^2)^\alpha ((\mathbf{q}_1 + \mathbf{q}_2)^2)^\alpha (r + q_2^2)^\beta}, \quad (15)$$

$$f_3(\alpha, \beta, \gamma, q_1) = \int_{q_2} \frac{1}{(q_2^2)^\alpha (r + q_2^2)^\beta (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\gamma}, \quad (16)$$

$$f_4(\alpha, \beta, \gamma, q_1) = \int_{q_2} \frac{1}{(q_2^2)^\alpha ((\mathbf{q}_1 + \mathbf{q}_2)^2)^\alpha (r + q_2^2)^\beta (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\gamma}. \quad (17)$$

In (14)-(17) values of integer parameters γ and β equal to 1 or 2, while parameter $\alpha = (d - a)/2$ or $\alpha = (d - a)$. We are interested in the case $a < d$, for which correlations of defects are relevant. Then the value of α is positive, that justifies form of (14)-(17). Therefore we have integrals with massive and massless propagators.

To obtain expressions for above functions (14)-(17) we appeal to the method of evaluating massive Feynman integrals based on the representation of massive denominators in the form of the Mellin-Barnes contour integrals. This method was developed in Refs.⁵⁵⁻⁵⁷ for evaluation of similar integrals.

First integral f_1 can be simply obtained by several method. We present its calculation along the lines of Ref.⁵⁵ in the Appendix B. The result reads:

$$f_1(\alpha, \beta, \vec{q}_1) = (r)^{\frac{d}{2}-\alpha-\beta} \frac{S_d \Gamma(\alpha + \beta - \frac{d}{2}) \Gamma(\frac{d}{2} - \alpha)}{2\Gamma(\beta)} {}_2F_1 \left[\begin{matrix} \alpha, \alpha + \beta - \frac{d}{2} \\ \frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right], \quad (18)$$

where $\Gamma(x)$ is the Gamma function, $S_d = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}$, ${}_2F_1$ is the Gauss hypergeometric function⁵⁸. For definition of ${}_2F_1$ and its integral representation see (A5) and (A6) in Appendix A. Note, that results for usual ϕ^4 theory are obtained as a particular case of our results at $a = d$ ($\alpha = 0$ in (18)). At $\alpha = 0$ function ${}_2F_1$ in (18) becomes unity and we come to known result (see e.g.²)

The obtained result can be used for calculation of $f_2(\alpha, \beta, q_1)$. To do this we transform (15) to the form similar to $f_1(\alpha, \beta, q_2)$ with the help of Feynman parametrisation (see B4).

$$f_2(\alpha, \beta, q_1) = \frac{\Gamma(\beta + \alpha)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx x^{\beta-1} (1-x)^{\alpha-1} \int_{q_2} \frac{1}{(q_2^2)^\alpha (rx + (\mathbf{q}_1 + \mathbf{q}_2)^2)^{\beta+\alpha}}. \quad (19)$$

Substituting result (18) into (19) instead the last integral we have

$$f_2(\alpha, \beta, q_1) = \frac{S_d \Gamma(\beta + 2\alpha - d/2) \Gamma(d/2 - \alpha)}{2\Gamma(\alpha)\Gamma(\beta)} \times \int_0^1 dx x^{\beta-1} (1-x)^{\alpha-1} (rx)^{d/2-2\alpha-\beta} {}_2F_1 \left[\begin{matrix} \alpha, \beta + 2\alpha - d/2 \\ d/2 \end{matrix} \middle| -\frac{q_1^2}{rx} \right]. \quad (20)$$

To calculate the integral we first use the Mellin-Barnes representation for the function ${}_2F_1$ (see (A7) in Appendix A). Then, performing an integration over x we get

$$f_2(\alpha, \beta, q_1) = r^{d/2-2\alpha-\beta} \frac{S_d \Gamma(d/2) \Gamma(d/2 - \alpha)}{2\Gamma(\alpha)\Gamma(\beta)} \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s) \Gamma(\alpha + s) \Gamma(d/2 - 2\alpha - s) \Gamma(2\alpha + \beta + s - d/2)}{\Gamma(d/2 - \alpha - s) \Gamma(s + d/2)} \left(\frac{q_1^2}{r} \right)^s. \quad (21)$$

Considering the integral over s , one can see that in the right half-plane of the complex variable s there are two series of poles due to $\Gamma(-s)$ and $\Gamma(d/2-2\alpha-s)$. Therefore integration can be performed with the help of residue theorem and the result is the following:

$$f_2(\alpha, \beta, q_1) = \frac{r^{d/2-2\alpha-\beta} S_d \Gamma(d/2) \Gamma(d/2-\alpha)}{2\Gamma(\beta) \Gamma(\alpha)} \times \left\{ \sum_{n \geq 0} \left(-\frac{q_1^2}{r} \right)^n \frac{1}{n!} \frac{\Gamma(\alpha+n) \Gamma(d/2-2\alpha-n) \Gamma(2\alpha+\beta+n-d/2)}{\Gamma(d/2-\alpha-n) \Gamma(n+d/2)} + \left(\frac{q_1^2}{r} \right)^{d/2-2\alpha} \sum_{n \geq 0} \left(-\frac{q_1^2}{r} \right)^n \frac{1}{n!} \frac{\Gamma(\beta+n) \Gamma(d/2-\alpha+n) \Gamma(2\alpha-n-d/2)}{\Gamma(\alpha-n) \Gamma(d-2\alpha+n)} \right\}. \quad (22)$$

Here, we can use formula (A3) of the Appendix A and finally we obtain the function $f_2(\alpha, \beta, q_1)$ in the form:

$$f_2(\alpha, \beta, q_1) = r^{\frac{d}{2}-2\alpha-\beta} \frac{S_d \Gamma(\frac{d}{2}-2\alpha) \Gamma(\beta+2\alpha-\frac{d}{2})}{2\Gamma(\beta)} {}_3F_2 \left[\begin{matrix} \beta+2\alpha-\frac{d}{2}, 1+\alpha-\frac{d}{2}, \alpha \\ 1+2\alpha-\frac{d}{2}, \frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right] + r^{-\beta} (q_1^2)^{\frac{d}{2}-2\alpha} \frac{S_d \Gamma(\frac{d}{2}) \Gamma(-\frac{d}{2}+2\alpha) \Gamma^2(\frac{d}{2}-\alpha)}{2\Gamma^2(\alpha) \Gamma(d-2\alpha)} {}_3F_2 \left[\begin{matrix} \frac{d}{2}-\alpha, \beta, 1-\alpha \\ d-2\alpha, 1-2\alpha+\frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right], \quad (23)$$

where functions ${}_3F_2$ are the generalised hypergeometric functions⁵⁸, defined in (A4) of Appendix A. Note that the second term disappears at $\alpha = 0$ and the function ${}_3F_2$ in first term is equal to unity. Therefore again the result of usual $O(N)$ -symmetric ϕ^4 theory² is recovered.

Taking in mind to calculate integral (8) we should consider the case $\beta = 2$:

$$f_2(\alpha, 2, q_1) = r^{d/2-2\alpha-2} \frac{S_d \Gamma(\frac{d}{2}-2\alpha) \Gamma(2+2\alpha-\frac{d}{2})}{2} {}_3F_2 \left[\begin{matrix} 2+2\alpha-\frac{d}{2}, 1+\alpha-\frac{d}{2}, \alpha \\ 1+2\alpha-\frac{d}{2}, \frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right] + r^{-2} (q_1^2)^{\frac{d}{2}-2\alpha} \frac{S_d \Gamma(\frac{d}{2}) \Gamma(-\frac{d}{2}+2\alpha) \Gamma^2(\frac{d}{2}-\alpha)}{2\Gamma^2(\alpha) \Gamma(d-2\alpha)} {}_3F_2 \left[\begin{matrix} \frac{d}{2}-\alpha, 2, 1-\alpha \\ d-2\alpha, 1-2\alpha+\frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right]. \quad (24)$$

In the first function ${}_3F_2$ the first numerator parameter is equal to the first denominator parameter plus one (for definition of numerator and denominator parameters see (A4)), therefore using definition for generalised hypergeometric function this function can be rewritten

ten as the sum of two ${}_2F_1$ functions. As final expression for $f_2(\alpha, 2, q_2)$ we have:

$$f_2(\alpha, 2, q_1) = r^{d/2-2\alpha-2} \frac{S_d \Gamma(\frac{d}{2}-2\alpha) \Gamma(2+2\alpha-\frac{d}{2})}{2} \left({}_2F_1 \left[\begin{matrix} 1+\alpha-\frac{d}{2}, \alpha \\ \frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right] - \right. \\ \left. \frac{q_1^2}{r} \frac{\alpha(1+\alpha-\frac{d}{2})}{(1+2\alpha-\frac{d}{2})\frac{d}{2}} {}_2F_1 \left[\begin{matrix} 2+\alpha-\frac{d}{2}, 1+\alpha \\ 1+\frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right] \right) + \\ r^{-2} (q_1^2)^{\frac{d}{2}-2\alpha} \frac{S_d \Gamma(\frac{d}{2}) \Gamma(-\frac{d}{2}+2\alpha) \Gamma^2(\frac{d}{2}-\alpha)}{2\Gamma^2(\alpha) \Gamma(d-2\alpha)} {}_3F_2 \left[\begin{matrix} \frac{d}{2}-\alpha, 2, 1-\alpha \\ d-2\alpha, 1-2\alpha+\frac{d}{2} \end{matrix} \middle| -\frac{q_1^2}{r} \right]. \quad (25)$$

Here, similarly as for function $f_1(\alpha, \beta, q_1)$ a particular case $\alpha = 0$ gives the known result, because the first ${}_2F_1$ function becomes unity and the two last terms in (25) disappear.

Note that function $f_2(\alpha, \beta, q_1)$ is a particular case of integral (25) of Ref.⁵⁵ with two different external momenta, when one of them is zero. In Ref.⁵⁵ result was obtained as a combination of two Lauricella generalized functions of three variables. When one puts one external momentum to be zero that result can be reduced to Kampé de Fériet functions⁵⁹ of two variables. We obtain a simpler expression in a form of a combination of two generalized hypergeometric functions ${}_3F_2$. Therefore our result (25) can be used to find possible reductions for more general hypergeometric functions.

Let us perform now the calculation of function $f_3(\alpha, \beta, \gamma, q_1)$. First with the help of Feynman parametrisation (B4) we get an integral with massive propagators only:

$$f_3(\alpha, \beta, \gamma, q_1) = \frac{\Gamma(\beta + \alpha)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx x^{\beta-1} (1-x)^{\alpha-1} \int_{q_2} \frac{1}{(rx + q^2)^{\beta+\alpha} (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\gamma}. \quad (26)$$

Then we can use expression (20) of Ref.⁵⁵ for the integral with two propagators having different masses. In our notation this formula has following form:

$$\int_q \frac{1}{(m_1^2 + q^2)^a (m_2^2 + (\mathbf{k} + \mathbf{q})^2)^b} = \frac{S_d \Gamma(\frac{d}{2}) \Gamma(a-\frac{d}{2})}{2\Gamma(a)} (m_1^2)^{d/2-a-b} \left\{ \frac{\Gamma(a) \Gamma(d/2-a) \Gamma(a+b-d/2)}{\Gamma(a-d/2) \Gamma(d/2) \Gamma(b)} \times \right. \\ \left. F_4 \left[a, a+b-d/2; d/2, a-d/2+1 \middle| -\frac{k^2}{m_2^2}, \frac{m_1^2}{m_2^2} \right] + \right. \\ \left. \left(\frac{m_1^2}{m_2^2} \right)^{d/2-a} F_4 \left[b, d/2; d/2, d/2-a+1 \middle| -\frac{k^2}{m_2^2}, \frac{m_1^2}{m_2^2} \right] \right\}, \quad (27)$$

where functions F_4 are hypergeometric Appell functions of two variables⁵⁸. Definition of Appell function F_4 is given in (A9) the Appendix A. Comparing this formula with the last integral in (26) one can see that in our case $m_1^2 = rx$, $m_2^2 = r$, $a = 1 + \alpha$ and $b = 1$.

Substituting (27) into (26) we get:

$$f_3(\alpha, \beta, \gamma, q_1) = \frac{S_d \Gamma(d/2) \Gamma(\beta + \alpha - d/2)}{2\Gamma(\alpha) \Gamma(\beta)} r^{d/2 - \alpha - \beta - \gamma} \left\{ \frac{\Gamma(\beta + \alpha) \Gamma(d/2 - \alpha - \beta) \Gamma(\gamma + \beta + \alpha - d/2)}{\Gamma(\beta + \alpha - d/2) \Gamma(d/2) \Gamma(\gamma)} \times \right. \\ \left. \int_0^1 dx x^{\beta-1} (1-x)^{\alpha-1} F_4 \left[\beta + \alpha, \beta + \gamma + \alpha - d/2; d/2, 1 + \beta + \alpha - d/2 \mid -\frac{q_1^2}{r}, x \right] + \right. \\ \left. \int_0^1 dx (1-x)^{\alpha-1} x^{d/2 - \alpha - 1} F_4 \left[\gamma, d/2; d/2, d/2 - \alpha - \beta + 1 \mid -\frac{q_1^2}{r}, x \right] \right\}. \quad (28)$$

Therefore we need to perform now only the integration over the Feynman parameter x . Using the definition of the Appell function F_4 we can rewrite it as an infinite sum of functions ${}_2F_1$:

$$F_4[a, b; c, c'; x, y] = \sum_{n \geq 0} \frac{x^n}{n!} \frac{(a)_n (b)_n}{(c)_n} {}_2F_1 \left[\begin{matrix} a + n, b + n \\ c' \end{matrix} \middle| y \right]. \quad (29)$$

Then substituting this expression into (28) we are able to use table integral for ${}_2F_1$ functions.

As a consequence, using formula 7.512.5 from⁶⁰ we obtain result in the form:

$$f_3(\alpha, \beta, \gamma, q_1) = \frac{S_d \Gamma(d/2)}{2} r^{d/2 - \alpha - \beta - \gamma} \sum_{n \geq 0} \left(-\frac{q_1^2}{r} \right)^n \frac{1}{n!} \left\{ \frac{\Gamma(d/2 - \alpha - \beta) \Gamma(\beta + \gamma + \alpha - d/2)}{\Gamma(d/2) \Gamma(\gamma)} \times \right. \\ \frac{(\beta + \alpha)_n (\beta + \gamma + \alpha - d/2)_n}{(d/2)_n} {}_3F_2 \left[\begin{matrix} \beta + \alpha + n, \beta + \gamma + \alpha - d/2 + n, \beta \\ 1 + \beta + \alpha - d/2, \beta + \alpha \end{matrix} \middle| 1 \right] + \\ \left. \frac{\Gamma(\beta + \alpha - d/2) \Gamma(d/2 - \alpha)}{\Gamma(d/2)} (\gamma)_n {}_3F_2 \left[\begin{matrix} \gamma + n, d/2 + n, d/2 - \alpha \\ d/2 - \alpha - \beta + 1, d/2 \end{matrix} \middle| 1 \right] \right\}. \quad (30)$$

We want to note here that the function $f_3(\alpha, \beta, \gamma, q_1)$ is a particular case of integral (35) of Ref.⁵⁵ with two different external momenta, when one of them is zero. Result of evaluation of such integral with both non-zero external momenta is the complex Lauricella generalized function⁵⁵, which can be reduced to Kampé de Fériet function when one of those momenta is put to be zero. Both arguments of obtained Kampé de Fériet function are not unit. Here we obtain result as two infinite sums of ${}_3F_2$ functions with unit argument, which makes possible to use known relations for ${}_3F_2$ functions with unit argument.

Let us consider the case $\beta = \gamma = 1$, which we need for integrals (7), (10), (11). For these

values of parameters we have:

$$f_3(\alpha, 1, 1, q_1) = \frac{S_d \Gamma(d/2)}{2} r^{d/2-\alpha-2} \left\{ \frac{\Gamma(d/2 - \alpha - 1) \Gamma(2 + \alpha - d/2)}{\Gamma(d/2)} \times \right. \\ \left. \sum_{n>0} \left(-\frac{q_1^2}{r} \right)^n \frac{1}{n!} \frac{(1+\alpha)_n (2+\alpha-d/2)_n}{(d/2)_n} {}_3F_2 \left[\begin{matrix} 1+\alpha+n, 2+\alpha-d/2+n, 1 \\ 2+\alpha-d/2, 1+\alpha \end{matrix} \middle| 1 \right] + \right. \\ \left. \Gamma(1+\alpha-d/2) \Gamma(d/2-\alpha) \sum_{n>0} \left(-\frac{q_1^2}{r} \right)^n \frac{\Gamma(1+n)}{n!} \frac{\Gamma(-1-2n)}{\Gamma(d/2-1-n) \Gamma(-n)} \right\}. \quad (31)$$

Consider first the ${}_3F_2$ function in the second line of (31). Using Thomae transformation for ${}_3F_2$ functions with unit argument (see e.g. formula 4.3.1 in⁶¹) we obtain:

$${}_3F_2 \left[\begin{matrix} 1+\alpha+n, 2+\alpha-d/2+n, 1 \\ 2+\alpha-d/2, 1-\alpha \end{matrix} \middle| 1 \right] = \frac{\Gamma(2+\alpha-d/2) \Gamma(1+\alpha) \Gamma(-1-2n)}{\Gamma(\alpha-n) \Gamma(1-d/2+\alpha-n)} \times \\ {}_3F_2 \left[\begin{matrix} \alpha, 1+\alpha-d/2, -1-2n \\ \alpha-n, 1+\alpha-d/2-n \end{matrix} \middle| 1 \right]. \quad (32)$$

Note, that in the right hand side of Eq. (32) the sum of the denominator parameters of ${}_3F_2$ function exceeds sum of its numerator parameters by one. It makes possible to use the Saalschutz's theorem for its transformation (see e.g. formula 2.3.1.4 in⁶¹):

$${}_3F_2 \left[\begin{matrix} \alpha, 1+\alpha-d/2, -1-2n \\ \alpha-n, 1+\alpha-d/2-n \end{matrix} \middle| 1 \right] = \frac{\Gamma(\alpha-n) \Gamma(d/2+n) \Gamma(1+n) \Gamma(-\alpha+d/2-1-n)}{\Gamma(\alpha-d/2+n) \Gamma(-n) \Gamma(d/2-1-n) \Gamma(\alpha+1+n)}. \quad (33)$$

Combining (31) with (32) and (33) and using for $\Gamma(-1-2n)$ the duplication formula (see (A2) in Appendix A) we get result for $f_3(q_1, 1, 1, \alpha)$ in the following form:

$$f_3(\alpha, 1, 1, q_1) = \frac{S_d \Gamma(\frac{d}{2})}{2} r^{\frac{d}{2}-\alpha-2} \left\{ -\frac{\Gamma(\frac{d}{2} - \alpha - 1)^2 \Gamma(2 + \alpha - \frac{d}{2})^2}{2\Gamma(\frac{d}{2} - 1) \Gamma(1 - \frac{d}{2} + \alpha) \Gamma(\frac{d}{2} - \alpha)^2} {}_2F_1 \left[\begin{matrix} 1, 2 - \frac{d}{2} \\ 3/2 \end{matrix} \middle| -\frac{q_1^2}{4r} \right] \right. \\ \left. - \frac{\Gamma(1 + \alpha - \frac{d}{2}) \Gamma(\frac{d}{2} - \alpha)}{2\Gamma(\frac{d}{2} - 1)} {}_2F_1 \left[\begin{matrix} 1, 2 - \frac{d}{2} \\ 3/2 \end{matrix} \middle| -\frac{q_1^2}{4r} \right] \right\} = \\ \frac{S_d (\frac{d}{2} - 1)}{2} \Gamma(\frac{d}{2} - \alpha - 1) \Gamma(2 + \alpha - \frac{d}{2}) r^{\frac{d}{2}-\alpha-2} {}_2F_1 \left[\begin{matrix} 1, 2 - \frac{d}{2} \\ 3/2 \end{matrix} \middle| -\frac{q_1^2}{4r} \right]. \quad (34)$$

At the evaluation of $f_4(\alpha, \beta, \gamma)$ we again can use result (18). To do that we apply Feynman parametrisation as well as use Mellin-Barnes representation for massive propagator,

presenting $f_4(\alpha, \beta, \gamma)$ in the form:

$$f_4(\alpha, \beta, \gamma, q_1) = \frac{1}{(2\pi i)} \frac{\Gamma(\gamma + \alpha)}{\Gamma(\gamma)\Gamma(\alpha)} \int_{-i\infty}^{i\infty} ds(r)^s \Gamma(-s) \Gamma(\beta + s) \times \int_0^1 dx \int_{q_2} \frac{x^{\gamma-1} (1-x)^{\alpha-1}}{(q_2^2)^{\beta+\alpha+s} (rx + (\mathbf{q}_1 + \mathbf{q}_2)^2)^{\gamma+\alpha}}. \quad (35)$$

Integration over q_1 gives the ${}_2F_1$ function. To make the integration over Feynman parameter x we rewrite it in the Mellin-Barnes representation. Then the integration over x results in:

$$f_4(\alpha, \beta, \gamma, q_1) = r^{d/2-2\alpha-\beta-\gamma} \frac{S_d \Gamma(d/2)}{2\Gamma(\gamma)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \int_{-i\infty}^{i\infty} dt \Gamma(-s) \Gamma(\beta + s) \Gamma(-t) \Gamma(\beta + \alpha + s + t) \times \frac{\Gamma(d/2 - \alpha - \beta - s) \Gamma(\beta + \gamma + 2\alpha - d/2 + s + t) \Gamma(-\beta - 2\alpha + d/2 - s - t)}{\Gamma(\alpha + \beta + s) \Gamma(d/2 - \beta - \alpha - s - t) \Gamma(d/2 + t)} \left(\frac{q_1^2}{r}\right)^t. \quad (36)$$

Closing the contour of integration to the right we use the set of the residues of the functions $\Gamma(-t)$ and $\Gamma(d/2 - 2\alpha - \beta - s - t)$ and obtain the result in the form:

$$f_4(\alpha, \beta, \gamma, q_1) = r^{d/2-2\alpha-\beta-\gamma} \frac{S_d \Gamma(d/2)}{2} \frac{1}{(2\pi i)} \times \int_{-i\infty}^{i\infty} ds \sum_{n \geq 0} \frac{1}{n!} \left\{ \left(-\frac{q_1^2}{r}\right)^n \Gamma(-s) \Gamma(\beta + s) \Gamma(\beta + \alpha + s + n) \Gamma(d/2 - 2\alpha - \beta - s) \times \frac{\Gamma(1 + \beta - d/2 + 2\alpha + s) \Gamma(1 + \beta - d/2 + \alpha + s + n) \Gamma(\beta + \gamma + 2\alpha - d/2 + s + n)}{\Gamma(\alpha + \beta + s) \Gamma(d/2 + n) \Gamma(1 + \beta - d/2 + \alpha + s) \Gamma(1 + \beta + 2\alpha - d/2 + s + n)} + \left(\frac{q_1^2}{r}\right)^{d/2-2\alpha-\beta-s+n} (-1)^n \frac{\Gamma(-s) \Gamma(\beta + s) \Gamma(d/2 - \alpha - \beta - s) \Gamma(\gamma + n)}{\Gamma(1 - \beta - 2\alpha + d/2 - s + n)} \times \frac{\Gamma(d/2 - \alpha + n) \Gamma(\beta + 2\alpha - d/2 + s) \Gamma(1 - \beta - 2\alpha + d/2 - s) \Gamma(1 - \alpha + n)}{\Gamma(\alpha + \beta + s) \Gamma(\alpha) \Gamma(1 - \alpha) \Gamma(d - 2\alpha - \beta - s + n)} \right\}. \quad (37)$$

Note that formula (A3) was used to obtain (37). Performing the next integration we close the contour to the right for the first term (using poles of $\Gamma(-s)$ and $\Gamma(d/2 - 2\alpha - \beta - s)$), while for the second term we close the contour to the left (using poles $\Gamma(\beta + s)$ and $\Gamma(-d/2 + 2\alpha + \beta + s)$). Then using formula (A3) our result can be presented via Kampé de Fériet functions (see

(A8) in Appendix A):

$$\begin{aligned}
f_4(\alpha, \beta, \gamma, q_1) = & \frac{r^{\frac{d}{2}-2\alpha-2} S_d \Gamma(\frac{d}{2})}{2} \left\{ \frac{\Gamma(\frac{d}{2}-2\alpha-\beta) \Gamma(\beta) \Gamma(\beta+\gamma+2\alpha-\frac{d}{2})}{\Gamma(d/2)} \times \right. \\
& F_{1,2,1}^{3,1,0} \left[\begin{matrix} 1+\beta+\alpha-\frac{d}{2}, \gamma+\beta+2\alpha-\frac{d}{2}, \beta+\alpha; \beta; 0 \\ 1+\beta+2\alpha-\frac{d}{2}; \beta+\alpha, 1+\beta+\alpha-\frac{d}{2}; \frac{d}{2} \end{matrix} \middle| 1, -\frac{q_1^2}{r} \right] + \\
& \frac{\Gamma(\beta-\frac{d}{2}+2\alpha) \Gamma(\gamma) \Gamma(\frac{d}{2}-2\alpha)}{\Gamma(d/2)} \\
& F_{1,3,1}^{3,2,0} \left[\begin{matrix} \gamma, -\alpha+\frac{d}{2}, 1-\alpha; 1, \frac{d}{2}-2\alpha; 0 \\ 1; 1-\alpha, -\alpha+\frac{d}{2}, 1-\beta+\frac{d}{2}-2\alpha; \frac{d}{2} \end{matrix} \middle| 1, -\frac{q_1^2}{r} \right] + \\
& \left(\frac{q_1^2}{r} \right)^{\frac{d}{2}-2\alpha} \frac{\Gamma(\beta) \Gamma(\gamma) \Gamma^2(\frac{d}{2}-\alpha) \Gamma(-\frac{d}{2}+2\alpha)}{\Gamma^2(\alpha) \Gamma(d-2\alpha)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; \beta, \frac{d}{2}-\alpha, 1-\alpha; \gamma, \frac{d}{2}-\alpha, 1-\alpha \\ 1-2\alpha+\frac{d}{2}, d-2\alpha; 0; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] + \\
& \frac{\Gamma(\frac{d}{2}-2\alpha) \Gamma(\gamma) \Gamma(\beta+2\alpha-\frac{d}{2})}{\Gamma(d/2)} \times \\
& F_{2,1,0}^{0,4,3} \left[\begin{matrix} 0; 1, \alpha, 1+\alpha-\frac{d}{2}, \beta+2\alpha-\frac{d}{2}; \gamma, \frac{d}{2}-\alpha, 1-\alpha \\ 1, d/2; 1+2\alpha-\frac{d}{2}; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] \Big\}. \quad (38)
\end{aligned}$$

To calculate integral (13), we have to consider $\beta = \gamma = 1$, in this case expression (38) is

reduced to the form:

$$\begin{aligned}
f_4(\alpha, 1, 1, q_1) = & \frac{r^{d/2-2\alpha-2} S_d \Gamma(d/2)}{2} \left\{ \frac{\Gamma(\frac{d}{2}-2\alpha-1) \Gamma(2+2\alpha-\frac{d}{2})}{\Gamma(d/2)} \times \right. \\
& F_{0,2,1}^{2,1,0} \left[\begin{matrix} 2+\alpha-\frac{d}{2}, 1+\alpha; 1; 0 \\ 0; 1+\alpha, 2+\alpha-\frac{d}{2}; \frac{d}{2} \end{matrix} \middle| 1, -\frac{q_1^2}{r} \right] + \\
& \frac{\Gamma(1-d/2+2\alpha) \Gamma(d/2-2\alpha)}{\Gamma(d/2)} F_{0,2,1}^{2,1,0} \left[\begin{matrix} -\alpha+d/2, 1-\alpha; 1; 0 \\ 0; 1-\alpha, -\alpha+d/2; d/2 \end{matrix} \middle| 1, -\frac{q_1^2}{r} \right] + \\
& \left(\frac{q_1^2}{r} \right)^{\frac{d}{2}-2\alpha} \frac{\Gamma^2(\frac{d}{2}-\alpha) \Gamma(-\frac{d}{2}+2\alpha)}{\Gamma^2(\alpha) \Gamma(d-2\alpha)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \frac{d}{2}-\alpha, 1-\alpha; 1, \frac{d}{2}-\alpha, 1-\alpha \\ 1-2\alpha+\frac{d}{2}, d-2\alpha; 0; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] + \\
& \frac{\Gamma(d/2-2\alpha) \Gamma(1+2\alpha-d/2)}{\Gamma(d/2)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \alpha, 1+\alpha-d/2; 1, d/2-\alpha, 1-\alpha \\ 1, d/2; 0; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] \left. \right\}. \quad (39)
\end{aligned}$$

Here further reduction can be performed. The first two terms can be presented as an infinite sum of ${}_3F_2$ functions with unit argument. Using for these functions Thomae transformation (see e.g. formula 4.3.1 in⁶¹) and Saalschutz's theorem (see e.g. formula 2.3.1.4 in⁶¹) as well as formulae (A2) and (A3) from Appendix A finally we get the following result:

$$\begin{aligned}
f_4(\alpha, 1, 1, q_1) = & \frac{r^{\frac{d}{2}-2\alpha-2} S_d \Gamma(\frac{d}{2})}{2} \left\{ \frac{(\frac{d}{2}-1) \Gamma(\frac{d}{2}-2\alpha-1) \Gamma(2+2\alpha-\frac{d}{2})}{\Gamma(\frac{d}{2})} {}_2F_1 \left[\begin{matrix} 1, 2-\frac{d}{2} \\ 3/2 \end{matrix} \middle| -\frac{q_1^2}{4r} \right] + \right. \\
& \left(\frac{q_1^2}{r} \right)^{\frac{d}{2}-2\alpha} \frac{\Gamma^2(\frac{d}{2}-\alpha) \Gamma(-\frac{d}{2}+2\alpha)}{\Gamma^2(\alpha) \Gamma(d-2\alpha)} F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \frac{d}{2}-\alpha, 1-\alpha; 1, \frac{d}{2}-\alpha, 1-\alpha \\ 1-2\alpha+\frac{d}{2}, d-2\alpha; 0; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] + \\
& \frac{\Gamma(\frac{d}{2}-2\alpha) \Gamma(1+2\alpha-\frac{d}{2})}{\Gamma(d/2)} F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \alpha, 1+\alpha-\frac{d}{2}; 1, \frac{d}{2}-\alpha, 1-\alpha \\ 1, \frac{d}{2}; 0; 0 \end{matrix} \middle| -\frac{q_1^2}{r}, -\frac{q_1^2}{r} \right] \left. \right\}. \quad (40)
\end{aligned}$$

Expressions (18), (24), (34), (40) obtained in this section for functions (14)-(17) will be further used for the calculation of the two-loop integrals. It is done in the following section.

IV. EXPRESSIONS FOR TWO-LOOP INTEGRALS

After performing an integration over q_2 in integrals(5)-(6), expressions for them have only one momentum integration over q_1 . Substituting functions (14)-(17) into (5)-(6) these

integrals can be written in the following form:

$$I(1, 1, 0, 2, 0, 2) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_1 \left(\frac{d-a}{2}, 2, q_1^2 \right), \quad (41)$$

$$I(1, 1, 0, 3, 0, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^3} f_1 \left(\frac{d-a}{2}, 1, q_1^2 \right), \quad (42)$$

$$I(1, 1, 0, 2, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_3 \left(\frac{d-a}{2}, 1, 1, q_1^2 \right), \quad (43)$$

$$I(1, 1, 1, 2, 0, 2) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_2 \left(\frac{d-a}{2}, 2, q_1^2 \right), \quad (44)$$

$$I(2, 1, 0, 3, 0, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{a-d}}{(r + q_1^2)^3} f_1 \left(\frac{d-a}{2}, 1, q_1^2 \right), \quad (45)$$

$$I(2, 1, 0, 2, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{a-d}}{(r + q_1^2)^2} f_3 \left(\frac{d-a}{2}, 1, 1, q_1^2 \right), \quad (46)$$

$$I(1, 2, 0, 2, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_3 \left(d-a, 1, 1, q_1^2 \right), \quad (47)$$

$$I(1, 1, 0, 1, 1, 2) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_3 \left(\frac{d-a}{2}, 1, 2, q_1^2 \right), \quad (48)$$

$$I(1, 1, 1, 2, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)^2} f_4 \left(\frac{d-a}{2}, 1, 1, q_1^2 \right). \quad (49)$$

Integrands in the integrals presented above are functions of absolute value of q_1 therefore integration over angular part can be performed separately. For the integrals including ${}_2F_1$ (that is with functions (14) and (15)) the table integrals can be used (see e. g. 2.21.1.15 in⁶²). Therefore we get the following expressions for (5)-(7), (9)-(11):

$$I(1, 1, 0, 2, 0, 2) = r^{a-4} \left(\frac{S_d}{2} \right)^2 \left\{ \frac{\Gamma(\frac{a}{2})\Gamma(\frac{d}{2})\Gamma^2(2 - \frac{a}{2})\Gamma(a-2)}{\Gamma(\frac{d+a}{2} - 2)} \left(1 + \frac{(2 - \frac{a}{2})(\frac{a-d}{2})}{(\frac{a}{2} - 1)(3-a)} \right) + \right. \\ \left. \frac{\Gamma(\frac{a}{2})\Gamma(\frac{d}{2})\Gamma(2 - \frac{2a-d}{2})\Gamma(4-a)\Gamma(\frac{a}{2} - 2)}{\Gamma(\frac{d-a}{2})\Gamma(2 - \frac{a-d}{2})} {}_3F_2 \left[\begin{matrix} 2 - \frac{2a-d}{2}, 4-a, 2 \\ 2 - \frac{a-d}{2}, 3 - \frac{a}{2} \end{matrix} \middle| 1 \right] \right\}, \quad (50)$$

$$I(1, 1, 0, 3, 0, 1) = r^{a-4} \left(\frac{S_d}{2} \right)^2 \left\{ \frac{\Gamma(\frac{a}{2})\Gamma(\frac{d}{2})\Gamma(1 - \frac{a}{2})\Gamma(3 - \frac{a}{2})\Gamma(a-1)}{2\Gamma(\frac{d+a}{2} - 1)} \times \right. \\ \left(1 + \frac{\frac{d-a}{2}}{(a-2)} \left(2 \frac{(1 - \frac{a}{2})}{(\frac{a}{2} - 2)} + \frac{(1 - \frac{a-d}{2})}{(a-3)} \right) \right) + \\ \left. \frac{\Gamma(\frac{a}{2})\Gamma(\frac{d}{2})\Gamma(3 - \frac{2a-d}{2})\Gamma(4-a)\Gamma(\frac{a}{2} - 3)}{\Gamma(\frac{d-a}{2})\Gamma(3 - \frac{a-d}{2})} {}_3F_2 \left[\begin{matrix} 3 - \frac{2a-d}{2}, 4-a, 3 \\ 3 - \frac{a-d}{2}, 4 - \frac{a}{2} \end{matrix} \middle| 1 \right] \right\}, \quad (51)$$

$$\begin{aligned}
I(1, 1, 0, 2, 1, 1) = & r^{a-4} \left(\frac{S_d}{2} \right)^2 \left\{ \left(\frac{d}{2} - 1 \right) \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{a}{2} - 1\right) \Gamma^2\left(2 - \frac{a}{2}\right) \left({}_2F_1 \left[1, 2 - \frac{d}{2}, \frac{3}{2}; \frac{1}{4} \right] + \right. \right. \\
& \left. \frac{(2 - \frac{d}{2})}{6(\frac{a}{2} - 1)} {}_2F_1 \left[2, 3 - \frac{d}{2}, \frac{5}{2}; \frac{1}{4} \right] \right) + \\
& 2^{a-4} \frac{(\frac{d}{2} - 1) \Gamma(\frac{a}{2} - 1) \Gamma(2 - \frac{a}{2}) \Gamma(\frac{3}{2}) \Gamma(3 - \frac{a}{2}) \Gamma(4 - \frac{a+d}{2}) \Gamma(\frac{a}{2} - 2)}{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{7-a}{2})} \times \\
& \left. {}_2F_1 \left[2, 4 - \frac{a+d}{2}, \frac{7-a}{2}; \frac{1}{4} \right] \right\}, \tag{52}
\end{aligned}$$

$$\begin{aligned}
I(2, 1, 0, 3, 0, 1) = & r^{\frac{3a-d}{2}-4} \left(\frac{S_d}{2} \right)^2 \left\{ \frac{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{a}{2}) \Gamma(\frac{2a-d}{2}) \Gamma(3 - \frac{2a-d}{2}) \Gamma(a-1)}{2\Gamma(\frac{d+a}{2}-1)} \times \right. \\
& \left(1 + \frac{\frac{d-a}{2}(1 - \frac{a}{2})}{(a-2)(\frac{2a-d}{2}-2)} \left(2 + \frac{(1 - \frac{a-d}{2})(2 - \frac{a}{2})}{(a-3)(\frac{2a-d}{2}-1)} \right) \right) + \\
& + \frac{\Gamma(\frac{a}{2}) \Gamma(\frac{d}{2}) \Gamma(3 - \frac{3a-2d}{2}) \Gamma(4 - \frac{3a-d}{2}) \Gamma(\frac{2a-d}{2} - 3)}{\Gamma(\frac{d-a}{2}) \Gamma(3 - (a-d))} \times \\
& \left. {}_3F_2 \left[\begin{matrix} 3 - \frac{3a-2d}{2}, 4 - \frac{3a-d}{2}, 3 \\ 3 - (a-d), 4 - \frac{2a-d}{2} \end{matrix} \middle| 1 \right] \right\}, \tag{53}
\end{aligned}$$

$$\begin{aligned}
I(2, 1, 0, 2, 1, 1) = & r^{\frac{3a-d}{2}-4} \left(\frac{S_d}{2} \right)^2 \left\{ \left(\frac{d}{2} - 1 \right) \Gamma\left(\frac{a}{2} - 1\right) \Gamma\left(2 - \frac{a}{2}\right) \Gamma\left(\frac{2a-d}{2}\right) \Gamma\left(2 - \frac{2a-d}{2}\right) \times \right. \\
& \left({}_2F_1 \left[1, 2 - \frac{d}{2}, \frac{3}{2}; \frac{1}{4} \right] + \frac{(2 - \frac{d}{2})}{6(\frac{2a-d}{2} - 1)} {}_2F_1 \left[2, 3 - \frac{d}{2}, \frac{5}{2}; \frac{1}{4} \right] \right) + \\
& 2^{2a-d-4} \frac{(\frac{d}{2} - 1) \Gamma(\frac{a}{2} - 1) \Gamma(2 - \frac{a}{2}) \Gamma(\frac{3}{2}) \Gamma(3 - \frac{2a-d}{2}) \Gamma(4-a) \Gamma(\frac{2a-d}{2} - 2)}{\Gamma(2 - \frac{d}{2}) \Gamma(\frac{7-(2a-d)}{2})} \times \\
& \left. {}_2F_1 \left[2, 4-a, \frac{7-(2a-d)}{2}; \frac{1}{4} \right] \right\}, \tag{54}
\end{aligned}$$

$$\begin{aligned}
I(1, 2, 0, 2, 1, 1) = & r^{\frac{3a-d}{2}-4} \left(\frac{S_d}{2} \right)^2 \left\{ \left(\frac{d}{2} - 1 \right) \Gamma\left(\frac{a}{2}\right) \Gamma\left(2 - \frac{a}{2}\right) \Gamma\left(\frac{2a-d}{2} - 1\right) \Gamma\left(2 - \frac{2a-d}{2}\right) \right. \\
& \left({}_2F_1 \left[1, 2 - \frac{d}{2}, \frac{3}{2}; \frac{1}{4} \right] + \frac{(2 - \frac{d}{2})}{6(\frac{a}{2} - 1)} {}_2F_1 \left[2, 3 - \frac{d}{2}, \frac{5}{2}; \frac{1}{4} \right] \right) + \\
& \frac{2^{2a-d-4} \frac{(\frac{d}{2}-1) \Gamma(\frac{2a-d}{2}-1) \Gamma(2-\frac{2a-d}{2}) \Gamma(\frac{3}{2}) \Gamma(3-\frac{a}{2}) \Gamma(4-\frac{a+d}{2}) \Gamma(\frac{a}{2}-2)}{\Gamma(2-\frac{d}{2}) \Gamma(\frac{7-a}{2})}} \times \\
& \left. {}_2F_1 \left[2, 4 - \frac{a+d}{2}, \frac{7-a}{2}; \frac{1}{4} \right] \right\}. \tag{55}
\end{aligned}$$

To get the integral $I(1, 1, 0, 1, 1, 2)$ (12) instead using formula (48) we can obtain it in other way. It is obvious that

$$I(1, 1, 0, 1, 1, 2) = -\frac{\partial}{\partial r} I(1, 1, 0, 1, 1, 1) - 2I(1, 1, 0, 2, 1, 1). \tag{56}$$

The last term in r.h.s. of (56) is already calculated (52), while the integral in the first term is expressed as $I(1, 1, 0, 1, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r+q_1^2)} f_3\left(\frac{d-a}{2}, 1, 1, q_1^2\right)$. Performing necessary calculation we get:

$$\begin{aligned}
I(1, 1, 0, 1, 1, 2) = & r^{a-4} (3-a) \left(\frac{S_d}{2} \right)^2 \left\{ \left(1 - \frac{d}{2} \right) \Gamma^2\left(\frac{a}{2}\right) \Gamma^2\left(1 - \frac{a}{2}\right) {}_2F_1 \left[1, 2 - \frac{d}{2}, \frac{3}{2}; \frac{1}{4} \right] + \right. \\
& \frac{2^{a-2} \frac{(\frac{d}{2}-1) \Gamma^2(\frac{a}{2}-1) \Gamma^2(2-\frac{a}{2}) \Gamma(\frac{3}{2}) \Gamma(3-\frac{a+d}{2})}{\Gamma(2-\frac{d}{2}) \Gamma(\frac{5-a}{2})}} {}_2F_1 \left[1, 3 - \frac{a+d}{2}, \frac{5-a}{2}; \frac{1}{4} \right] \left. \right\} - \\
& 2 \times \text{r.h.s. (52)} \tag{57}
\end{aligned}$$

To evaluate integral $I(1, 1, 1, 2, 0, 2)$, *i. e.* to integrate function (15), one needs to integrate the expressions that contains the hypergeometric function ${}_3F_2$. To do this one can represent this function as a integral of a Gauss ${}_2F_1$ function. It can be done *e. g.* using formula 2.21.1.4 of Ref.⁶². Then changing the order of integration for $I(1, 1, 1, 2, 0, 2)$ and

using formula 2.21.1.15 and 2.22.2.1 of Ref.⁶² one obtains the result in the form:

$$\begin{aligned}
I(1, 1, 1, 2, 0, 2) = & r^{\frac{3a-d}{2}-4} \left(\frac{S_d}{2} \right)^2 \left\{ \frac{\Gamma(\frac{2a-d}{2})\Gamma(2-\frac{2a-d}{2})\Gamma(\frac{d}{2})\Gamma(a-1)\Gamma(2-\frac{a}{2})}{\Gamma(\frac{d+a}{2}-1)} \times \right. \\
& \left[1 + \frac{\frac{a-d}{2}}{a-2} \left(1 + \frac{\frac{a}{2}}{(1-\frac{2a-d}{2})} \left(1 + \frac{(2-\frac{a}{2})(1-\frac{a-d}{2})}{\frac{a}{2}(a-3)} \right) \right) \right] + \\
& \frac{\Gamma(\frac{2a-d}{2})\Gamma(1-\frac{2a-d}{2})\Gamma(\frac{d}{2})\Gamma(3-a)\Gamma(1-\frac{2a-d}{2})\Gamma(\frac{a}{2}-2)}{\Gamma(1-\frac{a}{2})\Gamma(\frac{d-a}{2})\Gamma(2-\frac{a-d}{2})} \times \\
& \left[\left(1 - \frac{2a-d}{2} \right) {}_3F_2 \left[\begin{matrix} 2, 3-a, 2-\frac{2a-d}{2} \\ 2-\frac{a-d}{2}, 3-\frac{a}{2} \end{matrix} \middle| 1 \right] - \right. \\
& \left. \left(\frac{a}{2} - 2 \right) {}_3F_2 \left[\begin{matrix} 2, 3-a, 2-\frac{2a-d}{2} \\ 2-\frac{a-d}{2}, 2-\frac{a}{2} \end{matrix} \middle| 1 \right] \right] + \\
& \frac{\Gamma(\frac{3a-d}{2})\Gamma(2-\frac{3a-d}{2})\Gamma(\frac{d}{2})\Gamma^2(\frac{a}{2})\Gamma(\frac{d-2a}{2})}{\Gamma^2(\frac{d-a}{2})\Gamma(a)} \left[{}_3F_2 \left[\begin{matrix} 2, \frac{a}{2}, 1+\frac{a-d}{2} \\ a, 1+\frac{2a-d}{2} \end{matrix} \middle| 1 \right] + \right. \\
& \frac{1+\frac{a-d}{2}}{(1+\frac{2a-d}{2})(\frac{3a-d}{2}-1)} {}_3F_2 \left[\begin{matrix} 3, 1+\frac{a}{2}, 2+\frac{a-d}{2} \\ a, 1+\frac{2a-d}{2} \end{matrix} \middle| 1 \right] \left. + \right. \\
& \frac{\Gamma(\frac{d}{2})\Gamma(\frac{a}{2})\Gamma(\frac{d-2a}{2})\Gamma(1+\frac{2a-d}{2})\Gamma(\frac{3a-d}{2}-2)\Gamma(4-\frac{3a-d}{2})\Gamma(2-\frac{2a-d}{2})\Gamma(3-a)}{\Gamma^2(\frac{d-a}{2})\Gamma(1+\frac{a-d}{2})\Gamma(2-\frac{a-d}{2})\Gamma(3-\frac{a}{2})} \times \\
& \left[{}_3F_2 \left[\begin{matrix} 2, 3-a, 2-\frac{2a-d}{2} \\ 2-\frac{a-d}{2}, 3-\frac{a}{2} \end{matrix} \middle| 1 \right] + \right. \\
& \left. \frac{2(2-\frac{2a-d}{2})(3-a)}{(3-\frac{3a-d}{2})(3-\frac{a}{2})(2-\frac{a-d}{2})} {}_3F_2 \left[\begin{matrix} 3, 4-a, 3-\frac{2a-d}{2} \\ 3-\frac{a-d}{2}, 4-\frac{a}{2} \end{matrix} \middle| 1 \right] \right] \left. \right\}. \tag{58}
\end{aligned}$$

Result of calculation of the last integral $I(1,1,1,2,1,1)$ which includes integration of $f_4(\alpha,1,1,q_1)$ is rather lengthy, therefore we present it in Appendix C.

Obtained results can be also used within the in minimal subtraction RG scheme², extracting poles in $\epsilon = 4 - d$ and $\delta = 4 - a$. In this case we need to know the first terms in the double expansions in ϵ and δ for the corresponding hypergeometric functions. In the last decade a lot of efforts was put to derive ϵ -expansions for generalised hypergeometric functions analytically⁶³⁻⁷². Also algorithms for the same goal^{73,74} exploiting technique of nested sum were elaborated ready to use within a symbolic computer algebra system. They found their implementation in several different computer packages⁷⁵⁻⁷⁹. To be applied in our case all these methods should be extended to the case of double expansions.

V. CONCLUSIONS

In the present paper we have considered integrals appearing at the renormalization group analysis of the field-theoretical model including long-range correlated disorder. These integrals depend on the space dimension d and correlation parameter a . They were known analytically only in one-loop approximation at $d = 3$ as functions of a . Integrals of two-loop order were calculated only numerically for $d = 3$ and some set of values a in the range between 2 and 3^{38,39}. Here, we have evaluated two-loop integrals with untrivial massless propagators created by introduction of long-range correlated disorder for vertex function $\Gamma^{(4)}$ in form of combinations of hypergeometric functions with parameters dependent on d and a . The Mellin-Barnes method for evaluation of massive integrals was exploited at the calculations together with use of known results⁵⁵.

Main result of this paper is evaluation of one loop functions f_2 (15) and f_3 (16) for any parameters d and a in form of expressions with hypergeometric functions of one argument (24), (34). With the help of these expressions (18), (40) as well as with expressions for functions f_1 (14) and f_4 (17) two-loop integrals with untrivial massless propagators (5)-(13) were evaluated. Combining obtained results with double expansions in ϵ and δ of hypergeometric functions the poles in ϵ and δ can be found for these integrals. Therefore obtained results can be used at the renormalization-group study of different models of statistical physics that include defects having long range correlations. Our result also can be exploited for establishment of relationships between hypergeometric functions as it was done comparing results for Feynman diagrams evaluated by different methods⁸⁰.

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Appendix A: Some definitions

In this appendix we present definitions and some properties of functions used in the paper.

Gamma function is an extension of factorial function to noninteger and complex variables. It can be defined by the following expression (see e. g. 1.1.1 in⁵⁸):

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0. \quad (\text{A1})$$

Function $\Gamma(z)$ can be analytically continued for negative arguments, where it has series of poles for $z = -n$ at integer $n \geq 0$. For our calculation two properties of this function are useful. One from them is the duplication formula (see e. g. 1.2.15 in⁵⁸):

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z). \quad (\text{A2})$$

When the argument of the Γ function can be presented as $z - n$, where n is some natural number, the following formula can be used (see e. g. 1.2.3 in⁵⁸):

$$\Gamma(z - n) = (-1)^n \frac{\Gamma(z)\Gamma(1 - z)}{\Gamma(1 - z + n)}. \quad (\text{A3})$$

Definition of the **generalized hypergeometric series** of one variable is the following⁵⁸:

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{n! (b_1)_n (b_2)_n \dots (b_q)_n} z^n, \quad (\text{A4})$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is **Pochhammer symbol**. Parameters a_i in (A4) are called *numerator parameters* while b_i are called *denominator parameters*.

Gauss hypergeometric function can be defined as a particular case of (A4) at $p = 2$ and $q = 1$ ⁵⁸:

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| z \right] = {}_2F_1[a, b, c; z] = \sum_{n \geq 0} \frac{(a)_n (b)_n}{n! (c)_n} z^n. \quad (\text{A5})$$

Function ${}_2F_1$ has the following integral representation:

$${}_2F_1[a, b, c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{x^{b-1} (1-x)^{c-b-1}}{(1-xz)^a} dx, \quad (\text{A6})$$

and as a contour integral (Mellin-Barnes representation) in complex plane:

$${}_2F_1[a, b, c; z] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-\infty}^{i\infty} ds (-z)^s \Gamma(-s) \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}. \quad (\text{A7})$$

Extension of the theory of hypergeometric functions for the case of two variables gives in general case the **Kampé de Fériet functions**⁵⁹:

$$F_{C,D,D'}^{A,B,B'} \left[\begin{matrix} a_1, \dots, a_A; b_1, \dots, b_B; b'_1, \dots, b'_{B'} \\ c_1, \dots, c_C; d_1, \dots, d_D; d'_1, \dots, d'_{D'} \end{matrix} \middle| x, y \right] = \sum_{n \geq 0} \sum_{m \geq 0} \frac{(a_1)_{m+n} \dots (a_A)_{m+n} (b_1)_m \dots (b_B)_m (b'_1)_n \dots (b'_{B'})_n}{(c_1)_{m+n} \dots (c_C)_{m+n} (d_1)_m \dots (d_D)_m (d'_1)_n \dots (d'_{D'})_n} \frac{x^n y^m}{n! m!}. \quad (\text{A8})$$

Kampé de Fériet function $F_{0,1,1}^{2,0,0}$ corresponds to well known **Appell function of fourth type** F_4 ⁵⁸:

$$F_4[a, b; c, c' | x, y] = \sum_{n \geq 0} \sum_{m \geq 0} \frac{(a)_{m+n} (b)_{m+n}}{(c)_n (c')_m} \frac{x^n y^m}{n! m!}, \quad (\text{A9})$$

with the representation via Mellin-Barnes integral:

$$F_4[a, b; c, c' | x, y] = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)} \times \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \int_{-i\infty}^{i\infty} dt (-x)^s (-y)^t \Gamma(-s) \Gamma(-t) \frac{\Gamma(a+s+t) \Gamma(b+s+t)}{\Gamma(c+s) \Gamma(c'+t)}. \quad (\text{A10})$$

Appendix B: Mellin-Barnes method for massive integrals

In this appendix we apply the Mellin-Barnes transform to evaluate massive integrals (according to Ref.⁵⁵). The main idea consists in a usage of Mellin-Barnes representation of a function $1/(A+z)^\alpha$ via contour integral in the complex plane:

$$\frac{1}{(A+z)^\alpha} = \frac{1}{2\pi i \Gamma(\alpha)} \int_{-i\infty}^{i\infty} ds \frac{(z)^s}{A^{\alpha+s}} \Gamma(-s) \Gamma(\alpha+s), \quad (\text{B1})$$

where contour in the complex plane of s can be chosen to separate the “left” series of poles of the Γ functions in the integrand from the “right” poles. Therefore for evaluation of integral in (B1) the residue theorem can be used, where contour at infinity can be closed either in the right half-plane or in the left one in order to make the integrand decrease. Let us apply formula (B1) to the massive integral of function (14):

$$f_1(\alpha, \beta, q_1) = \int_{q_2} \frac{1}{(q_2^2)^\alpha (r + (\mathbf{q}_1 + \mathbf{q}_2)^2)^\beta}. \quad (\text{B2})$$

It gives us the following result:

$$f_1(\alpha, \beta, q_1) = \frac{1}{\Gamma(\beta) 2\pi i} \int_{-i\infty}^{i\infty} ds (r)^s \Gamma(-s) \Gamma(\beta+s) \int_{q_2} \frac{1}{(q_2^2)^\alpha ((\mathbf{q}_1 + \mathbf{q}_2)^2)^{\beta+s}}, \quad (\text{B3})$$

where the last integration over q_2 is performed only for massless propagators. It can be done with the help of Feynman parametrisation (for more details about Feynman parameters see e. g.^{1,2}):

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \frac{x^{a-1}(1-x)^{b-1}}{(xA + (1-x)B)^{a+b}}. \quad (\text{B4})$$

Then integral over q_2 in (B3) can be presented in the following form:

$$\int_{q_2} \frac{1}{(q_2^2)^a ((\mathbf{q}_2 + \mathbf{q}_1)^2)^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \int_{q_2} \frac{x^{b-1}(1-x)^{a-1}}{((\mathbf{q}_2 + x\mathbf{q}_1)^2 + x(1-x)q_1^2)^{a+b}}. \quad (\text{B5})$$

Here one can use substitution $\mathbf{q}_2 + x\mathbf{q}_1 \rightarrow \mathbf{q}'_2$. Then, in the integration over q'_2 the angular part can be evaluated separately (see (4)), for the rest formula 3.252.11 from⁶⁰ can be used.

Finally, after performing integration over x one gets result in form:

$$\int_{q_2} \frac{1}{(q_2^2)^a ((\vec{q}_1 + \vec{q}_2)^2)^b} = \frac{S_d}{2} \frac{\Gamma(d/2)\Gamma(\frac{d}{2}-a)\Gamma(d/2-b)\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)} (q_1^2)^{d/2-a-b}, \quad (\text{B6})$$

where $S_d = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}$. Substituting (B6) into (B3) one has

$$f_1(\alpha, \beta, q_1) = (q_1^2)^{d/2-\alpha-\beta} \frac{S_d \Gamma(d/2) \Gamma(d/2-\alpha)}{2\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s)\Gamma(d/2-\beta-s)\Gamma(\alpha+\beta+s-d/2)}{\Gamma(d-\alpha-\beta-s)} \left(\frac{r}{q_1^2}\right)^s. \quad (\text{B7})$$

Here, the following change of variables $s \rightarrow d/2 - \alpha - \beta - s$ is useful. It transforms the integral to the form

$$f_1(\alpha, \beta, q_1) = (r)^{d/2-\alpha-\beta} \frac{S_d \Gamma(d/2) \Gamma(d/2-\alpha)}{2\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s)\Gamma(\alpha+s)\Gamma(\alpha+\beta+s-d/2)}{\Gamma(d/2+s)} \left(\frac{q_1^2}{r}\right)^s. \quad (\text{B8})$$

Using definition (A7) one obtains

$$f_1(\alpha, \beta, q_1) = (r)^{d/2-\alpha-\beta} \frac{S_d \Gamma(\alpha+\beta-d/2) \Gamma(d/2-\alpha)}{2\Gamma(\beta)} {}_2F_1 \left[\begin{matrix} \alpha, \alpha+\beta-d/2 \\ d/2 \end{matrix} \middle| -\frac{q_1^2}{r} \right]. \quad (\text{B9})$$

Appendix C: Expression for integral $I(1, 1, 1, 2, 1, 1)$

Here, we present a way for calculation integral $I(1, 1, 1, 2, 1, 1)$. It can be done according to the formula (49). Also it is obvious that

$$I(1, 1, 1, 2, 1, 1) = -\frac{1}{3} \frac{\partial}{\partial r} I(1, 1, 1, 1, 1, 1), \quad (\text{C1})$$

where the integral in the r.h.s. of (C1) also is calculated via the same function $f_4(\frac{d-a}{2}, 1, 1, q_1^2)$:

$$I(1, 1, 1, 1, 1, 1) = \int_{q_1} dq_1 \frac{(q_1^2)^{\frac{a-d}{2}}}{(r + q_1^2)} f_4\left(\frac{d-a}{2}, 1, 1, q_1^2\right). \quad (C2)$$

Then we can substitute to (C2) the expression (40) and use Mellin-Barnes representation of the corresponding Kampé de Fériet functions in the expression for $f_4(\alpha, 1, 1, q_1)$. Obtained expression is quite long and cumbersome and we present it in the following form:

$$\begin{aligned} I(1, 1, 1, 1, 1, 1) = & r^{\frac{3a-d}{2}-4} \left(\frac{S_d}{2}\right)^2 \Gamma\left(\frac{d}{2}\right) \left\{ A(a, d) + B(a, d) + \right. \\ & \left. \left(\frac{d}{2}-1\right) \frac{\Gamma\left(2-a-\frac{d}{2}\right) \Gamma\left(-1+a-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left[\Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{a}{2}\right) {}_2F_1\left[\begin{matrix} 1, 2-\frac{d}{2} \\ \frac{3}{2} \end{matrix} \middle| \frac{1}{4}\right] + \right. \right. \\ & \left. \left. 2^{a-2} \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(3-\frac{a+d}{2}\right) \Gamma\left(2-\frac{a}{2}\right) \Gamma\left(\frac{a}{2}-1\right)}{\Gamma\left(2-\frac{d}{2}\right) \Gamma\left(\frac{5-a}{2}\right)} {}_2F_1\left[\begin{matrix} 1, 3-\frac{a+d}{2} \\ \frac{5-a}{2} \end{matrix} \middle| \frac{1}{4}\right] \right] \right\}, \quad (C3) \end{aligned}$$

where $A(a, d)$, $B(a, d)$ are results of integration of the second and the third terms respec-

tively:

$$\begin{aligned}
A(a, d) = & \frac{\Gamma\left(1+\frac{2a-d}{2}\right) \Gamma\left(\frac{3a-d}{2}\right) \Gamma\left(\frac{d-2a}{2}\right) \Gamma\left(1-\frac{3a-d}{2}\right) \Gamma^2\left(\frac{a}{2}\right)}{\Gamma^2\left(\frac{d-a}{2}\right) \Gamma(a) \Gamma\left(1+a-\frac{d}{2}\right)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \frac{a}{2}, 1+\frac{a-d}{2}; 1, \frac{a}{2}, 1+\frac{a-d}{2} \\ a, 1+a-\frac{d}{2}; 0; 0 \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma\left(1+a-\frac{d}{2}\right) \Gamma\left(\frac{3a-d}{2}-1\right) \Gamma\left(-a+\frac{d}{2}\right) \Gamma\left(2-\frac{3a-d}{2}\right) \Gamma\left(\frac{a}{2}\right) \Gamma(2-a) \Gamma\left(1-a+\frac{d}{2}\right)}{\Gamma^2\left(\frac{d-a}{2}\right) \Gamma\left(2-\frac{a}{2}\right) \Gamma\left(1+\frac{a-d}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \frac{a}{2}, 1+\frac{a-d}{2}; 1, 2-a, 1-a+\frac{d}{2} \\ 2-\frac{a}{2}, 1-\frac{a-d}{2}; 0; 0 \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma\left(\frac{a}{2}-1\right) \Gamma\left(\frac{a-d}{2}\right) \Gamma\left(1+\frac{2a-d}{2}\right) \Gamma\left(\frac{3a-d}{2}-2\right) \Gamma\left(-a+\frac{d}{2}\right) \Gamma\left(2-\frac{a-d}{2}\right) \Gamma\left(3-\frac{3a-d}{2}\right)}{\Gamma^2\left(1+\frac{a-d}{2}\right) \Gamma^2\left(\frac{d-a}{2}\right) \Gamma^2\left(1-\frac{a-d}{2}\right) \Gamma\left(2-\frac{a}{2}\right)} \times \\
& \Gamma(3-a) \Gamma\left(2-\frac{2a-d}{2}\right) F_{0,3,2}^{2,2,1} \left[\begin{matrix} 3-a, 2-a+\frac{d}{2}; 1, 3-\frac{3a-d}{2}; 1 \\ 0; 2-\frac{a}{2}, 1-\frac{a-d}{2}, 3+\frac{3a-5d}{2}; 2-\frac{a}{2}, 1-\frac{a-d}{2} \end{matrix} \middle| 1, 1 \right] + \\
& \frac{1}{2} \Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{a}{2}\right) \Gamma\left(2-\frac{d}{2}\right) {}_2F_1 \left[\begin{matrix} 1, 2-\frac{d}{2} \\ 3/2 \end{matrix} \middle| \frac{1}{4} \right] + \\
& \frac{\Gamma(a-2) \Gamma\left(1+a-\frac{d}{2}\right) \Gamma\left(-a+\frac{d}{2}\right) \Gamma(3-a) \Gamma\left(2-\frac{a}{2}\right) \Gamma\left(3-\frac{a+d}{2}\right)}{\Gamma\left(1+\frac{a-d}{2}\right) \Gamma\left(\frac{d-a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right) \Gamma\left(2-\frac{a}{2}\right)} \times \\
& F_{0,2,2}^{2,1,1} \left[\begin{matrix} 2-a, 3-\frac{a+d}{2}; 3-a; 1 \\ 0; 3+a-2d, 2-\frac{d}{2}; 2-\frac{a}{2}, 1-\frac{a-d}{2} \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma(2-a) \Gamma\left(1+a-\frac{d}{2}\right) \Gamma\left(\frac{3a-d}{2}-1\right) \Gamma\left(-a+\frac{d}{2}\right) \Gamma\left(1-a+\frac{d}{2}\right) \Gamma\left(\frac{a}{2}\right) \Gamma\left(2-\frac{3a-d}{2}\right)}{\Gamma\left(1+\frac{a-d}{2}\right) \Gamma^2\left(\frac{d-a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right) \Gamma\left(2-\frac{a}{2}\right)} \times \\
& F_{0,2,2}^{2,1,1} \left[\begin{matrix} \frac{a}{2}, 1+\frac{a-d}{2}; 1; 1 \\ 0; a-1, a-\frac{d}{2}; 2-\frac{a}{2}, 1-\frac{a-d}{2} \end{matrix} \middle| 1, 1 \right], \tag{C4}
\end{aligned}$$

$$\begin{aligned}
B(a, d) = & \frac{\Gamma\left(a-\frac{d}{2}\right) \Gamma\left(1-\frac{2a-d}{2}\right) \Gamma\left(\frac{a}{2}\right) \Gamma\left(1-\frac{a}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, \frac{a}{2}, 1+\frac{a-d}{2}; 1, 1-\frac{a}{2}, \frac{d-a}{2} \\ 1, \frac{d}{2}; 0; 0 \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma\left(\frac{a}{2}-1\right) \Gamma\left(a-\frac{d}{2}\right) \Gamma^2\left(1-a+\frac{d}{2}\right) \Gamma(2-a) \Gamma\left(1-\frac{a}{2}\right)}{\Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{d-a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right)} \times \\
& F_{2,0,0}^{0,3,3} \left[\begin{matrix} 0; 1, 2-a, 1-a+\frac{d}{2}; 1, \frac{a}{2}, 1+\frac{a-d}{2} \\ 2-\frac{a}{2}, 1-\frac{a-d}{2}; 0; 0 \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma\left(3-\frac{a}{2}\right) \Gamma\left(\frac{a}{2}-2\right) \Gamma\left(-\frac{a}{2}\right) \Gamma\left(a-\frac{d}{2}\right) \Gamma\left(1-a+\frac{d}{2}\right) \Gamma\left(\frac{d-a}{2}-1\right) \Gamma\left(3-\frac{d}{2}\right)}{\Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{d-a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right)} \times \\
& F_{0,2,2}^{2,1,1} \left[\begin{matrix} 2, 3-\frac{d}{2}; 1; 1 \\ 0; 1+\frac{a}{2}, 2+\frac{a-d}{2}; 2-\frac{a}{2}, 1-\frac{a-d}{2} \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma\left(2-\frac{d}{2}\right) \Gamma\left(-1+a-\frac{d}{2}\right) \Gamma\left(a-\frac{d}{2}\right) \Gamma\left(1-a+\frac{d}{2}\right) \Gamma\left(2-a+\frac{d}{2}\right)}{2\Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{a}{2}\right)} {}_2F_1 \left[\begin{matrix} 1, 2-\frac{d}{2} \\ 3/2 \end{matrix} \middle| \frac{1}{4} \right] + \\
& \frac{\Gamma(3-a) \Gamma(a-2) \Gamma\left(a-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}-1\right) \Gamma\left(1-a+\frac{d}{2}\right) \Gamma\left(2-\frac{a}{2}\right) \Gamma\left(3-\frac{a+d}{2}\right)}{\Gamma\left(1+\frac{a-d}{2}\right) \Gamma\left(\frac{d-a}{2}\right) \Gamma\left(2-\frac{a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right)} \times \\
& F_{0,2,1}^{2,1,0} \left[\begin{matrix} 2-\frac{a}{2}, 3-\frac{a+d}{2}; 1; 0 \\ 0; 2-\frac{a}{2}, 1-\frac{a-d}{2}; 2-\frac{d}{2} \end{matrix} \middle| 1, 1 \right] + \\
& \frac{\Gamma(2-a) \Gamma(a-1) \Gamma\left(a-\frac{d}{2}\right) \Gamma\left(\frac{a}{2}-1\right) \Gamma^2\left(1-a+\frac{d}{2}\right)}{\Gamma\left(1-\frac{a}{2}\right) \Gamma\left(\frac{d-a}{2}\right) \Gamma\left(1-\frac{a-d}{2}\right) \Gamma(a-1)} \times \\
& F_{0,2,2}^{2,1,1} \left[\begin{matrix} \frac{a}{2}, 1+\frac{a-d}{2}; 1; 1 \\ 0; a-1, a-\frac{d}{2}; 2-\frac{a}{2}, 1-\frac{a-d}{2} \end{matrix} \middle| 1, 1 \right] \tag{C5}
\end{aligned}$$

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